Exercise 3.3.14

(a) Consider a function $f(x)$ that is even around $x = L/2$. Show that the odd coefficients (n odd) of the Fourier cosine series of $f(x)$ on $0 \leq x \leq L$ are zero.

(b) Explain the result of part (a) by considering a Fourier cosine series of $f(x)$ on the interval $0 \leq x \leq L/2$.

Solution

The Fourier cosine series expansion of $f(x)$, a piecewise smooth function defined on $0 \leq x \leq L$, is given by

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},$$

where

$$\begin{aligned} A_0 &= \frac{1}{L} \int_0^L f(x)\,dx \\ A_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx. \end{aligned}$$

Replace x with $x + L/2$ to translate everything to the left by $L/2$ units.

$$\begin{aligned} A_0 &= \frac{1}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) dx \\ A_n &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \cos\left[\frac{n\pi}{L}\left(x + \frac{L}{2}\right)\right] dx. \end{aligned}$$

Consider the odd coefficients by setting $n = 2k + 1$.

$$\begin{aligned} A_{2k+1} &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \cos\left[\frac{(2k+1)\pi}{L}\left(x + \frac{L}{2}\right)\right] dx \\ &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \cos\left[\frac{(2k+1)\pi x}{L} + \frac{(2k+1)\pi}{2}\right] dx \\ &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \left[\cos \frac{(2k+1)\pi x}{L} \cos \frac{(2k+1)\pi}{2} - \sin \frac{(2k+1)\pi x}{L} \sin \frac{(2k+1)\pi}{2}\right] dx \\ &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \left[(0) \cos \frac{(2k+1)\pi x}{L} - (-1)^k \sin \frac{(2k+1)\pi x}{L}\right] dx \\ &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) (-1)^{k+1} \sin \frac{(2k+1)\pi x}{L}\,dx \\ &= \frac{2(-1)^{k+1}}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin \frac{(2k+1)\pi x}{L}\,dx \end{aligned}$$

Note that f is even, and sine is odd. The product of an even function and an odd function is odd, and the integral of an odd function over a symmetric interval is zero.

$$A_{2k+1} = \frac{2(-1)^{k+1}}{L}(0) = 0$$

Therefore, the odd coefficients in the Fourier cosine series expansion of $f(x)$ are zero if f is even with respect to $x = L/2$. As an example, consider the Fourier cosine series of $f(x)$ on the interval $0 \leq x \leq L/2$.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{\frac{L}{2}} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{2n\pi x}{L}$$

This series is the L-periodic even extension of $f(x)$ to the whole line $(-\infty < x < \infty)$, so it is even with respect to $x = L/2$. The point to note is that the series has no cosine terms with odd multiples of $\pi x/L$.